# Rainbow triangles in edge-colored graphs

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# Outline

# Rainbow cycles in edge-colored graphs

- The main topic of this talk is presenting sufficient conditions for rainbow triangles in edge-colored graphs.
- Let $G$ be an edge-colored graph. We call a subgraph $H$ of $G$ is rainbow if every two edges of $H$ are assigned with distinct colors.
- This talk is based on the following papers of mine and my coauthors.

- Li, Binlong; **Ning, Bo**; Xu, Chuandong; Zhang, Shenggui, Rainbow triangles in edge-colored graphs. European J. Combin. 36 (2014), 453–459.
- Li, Ruonan; **Ning, Bo**; Zhang, Shenggui, Color degree sum conditions for rainbow triangles in edge-colored graphs. Graphs Combin. 32 (2016), no. 5, 2001–2008.
- Fujita, Shinya; **Ning, Bo**; Xu, Chuandong; Zhang, Shenggui, On sufficient conditions for rainbow cycles in edge-colored graphs. Discrete Math. 342 (2019), no. 7, 1956–1965.
- Li Xueliang, **Ning Bo**, Shi Yongtang, Zhang Shenggui, Counting rainbow triangles in edge-colored graphs, arXiv:2112.14458.

- To find a tight sufficient condition for a rainbow cycle in an edge-colored graph is an interesting problem in the area of rainbow subgraphs of edge-colored graphs.
- It is lucky for us to have the following beautiful theorem due to Hao Li.

**Theorem (Hao Li, 2013)**

*Let $G$ be an edge-colored graph. If $\delta^c \geq \frac{n+1}{2}$, then $G$ contains a rainbow triangle.*

- Li's theorem is originally a conjecture proposed by Hao Li and Guanghui Wang in 2006.
- In their 2009 published paper (EUJC, 2009), they obtained three weaker results.
- They also constructed example to show that for any positive integer $D$, there exists a class of graph with minimum color degree at least $D$ but contains no rainbow cycle.

Binlong Li et al. in fact obtained two proofs of Li-Wang's conjecture.

**Theorem (Li, Ning, Xu and Zhang, 2014)**

*Let $G$ be an edge-colored graph on $n \geq 5$ vertices. If $\delta^c \geq \frac{n}{2}$, then $G$ contains a rainbow triangle unless $G$ is a properly-colored $K_{\frac{n}{2},\frac{n}{2}}$ where $n$ is even.*

**Theorem (Li, Ning, Xu and Zhang, 2014)**

*Let $G$ be an edge-colored graph on $n \geq 3$ vertices. If $\sum_{v \in V(G)} d^c(v) \geq \frac{(n+1)n}{2}$, then $G$ contains a rainbow triangle.*

Very interesting for us, Li et al. introduced a new type of sufficient condition for rainbow subgraphs in edge-colored graphs.

**Theorem (Li, Ning, Xu and Zhang, 2014)**

*Let $G$ be an edge-colored graph on $n \geq 3$ vertices. If $e(G) + c(G) \geq \frac{(n+1)n}{2}$, then $G$ contains a rainbow triangle.*

### Remark

*The term* $\frac{(n+1)n}{2} = \binom{n}{2} + n$ *here is meaningful. That is, we can write* $\frac{(n+1)n}{2}$ *as* $\binom{n}{2} + ar(n, K_3)$.

### Remark

*Xu et al. extended the theorem above to rainbow cliques (EUJC, 2016) and properly-colored* $C_4$ *(DM, 2020).*

**Theorem (Xu, Hu, Wang, Zhang, 2016)**

*Let $G$ be an edge-colored graph on $n$ vertices and $n \geq k \geq 4$. If*

$$e(G) + c(G) \geq \binom{n}{2} + t_{n,k-2} + 2,$$

*then $G$ contains a rainbow $K_k$.*

**Theorem (Xu, Magnant, Zhang, 2020)**

*Let $G$ be an edge-colored graph with order $n$. If*

$$e(G) + c(G) \geq \frac{n(n+1)}{2},$$

*then $G$ contains a properly colored $C_4$.*

Xu et al.'s result confirmed the following conjecture for the case $k = 3$.

**Conjecture (Manoussakis, Spyratos, Tuza, Voigt, 1996)**

*Let $G$ be an edge-colored complete graph with order $n$. For $n > k \geq 2$, if*

$$c(G) > \max\{n - k + \binom{k}{2}, \left\lfloor \frac{k}{3} \right\rfloor (n - \left\lfloor \frac{k}{3} \right\rfloor) + \binom{\lfloor \frac{k}{3} \rfloor}{3} + 1\},$$

*then $G$ contains a properly colored $C_{k+1}$.*

- The original purpose is to study the supersaturation problem of rainbow triangles in edge-colored graphs. This problem is obviously motivated by the study of supersaturation problem of triangles in graphs.
- It studies the following function: for triangle $C_3$ and for integers $n, t \geq 1$,
  $h_{C_3}(n,t) = min\{t(G) : |V(G)| = n, |E(G)| = ex(n, C_3) + t\}$,
  where $t(G)$ is the number of $C_3$ in $G$ and $ex(n, C_3)$ is the Turán function of $C_3$.

# History

- Improving Mantel's theorem, Rademacher (unpublished, 1955) proved that $h_{C_3}(n,1) \geq \lfloor \frac{n}{2} \rfloor$.
- Erdős (1962) proved that $h_{C_3}(n,k) \geq k\lfloor \frac{n}{2} \rfloor$ where $k \leq cn$ for some constant $c$.
- In fact, Erdős conjectured that $h_{C_3}(n,k) \geq k\lfloor \frac{n}{2} \rfloor$ for all $k < \lfloor \frac{n}{2} \rfloor$, which was finally resolved by Lovász and Simonovits (1983).

**Theorem (Lovász and Simonovits, 1983)**

*Let $G$ be a graph on $n$ vertices. If $e(G) \geq \frac{n^2}{4} + k$ where $k < \lfloor \frac{n}{2} \rfloor$, then $G$ contains at least $k\lfloor \frac{n}{2} \rfloor$ triangles.*

- One can ask for a rainbow analog of the above Erdős' conjecture. In this direction, Ehard and Mohr answered an open problem proposed by Fujita, Ning, Xu and Zhang (DM, 2018).
- If we consider $e(G) + c(G)$ as a variant of Turán function in edge-colored graphs, then the theorem above tells us that the supersaturation phenomenon of rainbow triangles under this type of condition is quite different from the original one.

**Theorem (Ehard, Mohr, 2020)**

*There are at least k rainbow triangles in an edge-colored graph $(G, C)$ such that $e(G) + c(G) \geq \binom{n+1}{2} + k - 1$.*

So it is natural for us to find a counting version of Hao Li's theorem.

**Theorem (X. Li, Ning, Shi, Zhang, Dec. 2021)**

*Let $(G, C)$ be an edge-colored graph on $n$ vertices. Suppose that $\delta^c(G) \geq \frac{n+1}{2}$, and subject to this, $e(G)$ is minimal. Then*

$$rt(G) \geq \frac{e(G)(\overline{\sigma}_2^c(G) - n)}{3} + \frac{1}{6} \sum_{v \in V(G)} (n - d(v) - 1)(d(v) - d^c(v)).$$

As consequences of the above Theorem, we obtain two counting versions of Hao Li's theorem.

**Theorem (X. Li, Ning, Shi, Zhang, Dec. 2021)**

*Let $(G, C)$ be an edge-colored graph on $n$ vertices. Then*

$$rt(G) \geq \frac{1}{6}\delta^c(G)(2\delta^c(G) - n)n.$$

*In particular, if $\delta^c(G) > cn$ for $c > \frac{1}{2}$, then*

$$rt(G) \geq \frac{c(2c-1)}{6}n^3.$$

One may wonder the tightness of above theorem The following example shows that it is the best possible.

**Example**

*Let $G$ be a rainbow $k$-partite Turán graph on $n$ vertices where $k|n$ and $k \geq 3$. Then there are exactly $\binom{k}{3}(\frac{n}{k})^3 = \frac{(k-1)(k-2)}{6k^2}n^3$ rainbow triangles. By our Theorem, there are at least $\binom{k}{3}(\frac{n}{k})^3 = \frac{(k-1)(k-2)}{6k^2}n^3$ rainbow triangles.*

Setting $\delta^c(G) = \frac{n+1}{2}$ in the Theorem, we obtain the right hand of the following.

**Theorem (X. Li, Ning, Shi, Zhang, Dec. 2021)**

*For even $n \geq 4$, we have $\frac{n^2}{4} \geq f(n) \geq \frac{n^2+2n}{6}$; for odd $n \geq 3$, we have $\frac{n^2-1}{8} \geq f(n) \geq \frac{n^2+n}{12}$.*

we infer $f(n) = \Theta(n^2)$.

## Example

*Let $(G, C)$ be a rainbow graph of order $n$ where $n$ is divisible by 4. Let $V(G) = X_1 \cup X_2$, $|X_1| = |X_2| = \frac{n}{2}$, and each of $G[X_1]$ and $G[X_2]$ consists of a perfect matching of size $\frac{n}{4}$. In addition, $G - E(X_1) - E(X_2)$ is balanced and complete bipartite. For each edge $e \in E(X_1)$, it is contained in exactly $\frac{n}{2}$ rainbow triangles. So does each edge in $G[X_2]$. Therefore, there are exactly $\frac{n^2}{4}$ rainbow triangles in $G$.*

## Example

*Let $(G, C)$ be a rainbow graph of order $n$ where $n \equiv 1 \pmod 4$. Let $V(G) = X_1 \cup X_2$, $|X_1| = \frac{n+1}{2}$ and $|X_2| = \frac{n-1}{2}$, and $G[X_1]$ consists of a perfect matching of size $\frac{n+1}{4}$. In addition, $G - E(X_1)$ is complete bipartite. For each edge $e \in E(X_1)$, it is contained in exactly $\frac{n-1}{2}$ rainbow triangles and so does each edge in $G[X_1]$. Therefore, there are exactly $\frac{n^2-1}{8}$ rainbow triangles in $G$.*

In 2005, Broersma, X. Li, Woeginger, and Zhang proved a sufficient condition for rainbow short cycles.

**Theorem (Broersma, X. Li, Woeginger, Zhang, 2005)**

*Let $G$ be an edge-colored graph on $n \geq 4$ vertices. If $|CN(u) \cup CN(v)| \geq n-1$ for every pair of vertices $u, v \in V(G)$, then $G$ contains a rainbow $C_3$ or a rainbow $C_4$.*

Broersma et al.'s theorem was generalized by Fujita, Ning, Xu and Zhang (DM, 2019) to the one forcing rainbow triangles under the same condition.

**Theorem (Fujita, Ning, Xu, and Zhang, 2019)**

*Let $G$ be an edge-colored graph on $n \geq 4$ vertices. If $|CN(u) \cup CN(v)| \geq n-1$ for every pair of vertices $u, v \in V(G)$, then $G$ contains a rainbow $C_3$, unless $G$ is a properly-colored $K_{\frac{n}{2},\frac{n}{2}}$ where $n$ is even.*

We extend both theorems mentioned to a counting version as follows.

**Theorem (X. Li, Ning, Shi, Zhang, Dec. 2021+)**

*Let $(G, C)$ be an edge-colored graph of order $n \geq 4$ such that $|CN(u) \cup CN(v)| \geq n$ for every pair of vertices $u, v \in V(G)$. Then $G$ contains $\frac{n^2-2n}{24}$ rainbow $C_3$'s.*

As a warm-up, we first give the sketch of the proof of Hao Li's theorem.

- Let $v$ be a vertex such that the maximum number, denoted by $f = f(v)$, of monochromatic edges incident with $v$ is maximum among all vertices of the graph.
- If $f = 1$, then $G$ is rainbow and so contains a rainbow triangle. So assume $f \geq 2$.
- assume that $S = \{u_1, u_2, \ldots, u_f\} \subset N(v)$ with $C(vu_i) = c_0$ for any $1 \leq i \leq f$. Choose a maximum rainbow-neighborhood of $v$, denoted by $T = \{x_1, x_2, x_3, \ldots, x_k\}$ such that $C(vx_i) = c_i$ and $c_i \neq c_0$ for $1 \leq i \leq k$. Thus, $c_0, c_1, c_2, \ldots, c_k$ are pairwise distinct.

- Since $G$ has no rainbow triangle, either $C(x_ix_j) = C(vx_i) = c_i$ or $C(x_ix_j) = C(vx_j) = c_j$ for any $x_ix_j \in E(G)$ with $1 \leq i \leq k$ and $1 \leq j \leq k$. Similarly, if there is an edge $u_ix_j$, its color is either $c_0$ or $c_j$ (can be improved!!).
- We define an oriented graph $D(T, A(D))$ as follows:
  $V(D) = T$ and
  $A(D) = \{x_ix_j : x_ix_j \in E(G[T]) \text{ and } C(x_ix_j) = c_j\}$
- for any $i$, $x_i$ has at least $d^c(x_i) - (d_D^+(x_i) + |\{c_i\}|)$ distinct neighbors in $G - (S \cup T \cup \{v\})$.

- we have

$$n \geq |S \cup T \cup \{v\}| + d^c(x_i) - (d_D^+(x_i) + 1),$$

  and it follows that $d_D^+(x_i) \geq f$.
- We also have $d_D^-(x_i) \leq f - 1$. A contradiction!

This completes the proof of Li's theorem.

Our proof of main counting result is based on two main lemmas.

**Lemma (X. Li, Ning, Shi, Zhang, Dec. 2021+)**

*Let $(G, C)$ be an edge-colored graph on $n$ vertices with $\delta^c(G) \geq \frac{n+1}{2}$ and furthermore, $e(G)$ is minimal. Then for each $v \in V(G)$, we have*

$$rt(G; v) \geq \frac{1}{2}((n - d(v) - 1)(d(v) - d^c(v)) + \sum_{1\leq j\leq d^c(v)} \sum_{a\in N_j(v)} (d_j(v) - d_j(a)) + \sum_{a\in N_v} (d^c(v) + d^c(a) - n)).$$

### Lemma (X. Li, Ning, Shi, Zhang, Dec. 2021+)

*Let $(G, C)$ be an edge-colored graph with vertex set $V(G)$ and $\delta^c(G) \geq \frac{n+1}{2}$, and furthermore, $e(G)$ is minimal. Let $I_v = \{C(uv) : uv \in E(G)\}$. For $k \in I_v$, $N_k(v) := \{u \in N_v : C(uv) = k\}$ and $d_k(v) := |N_k(v)|$. Then*

$$\sum_{v \in V(G)} \sum_{k \in I_v} \sum_{a \in N_k(v)} (d_k(v) - d_k(a)) = 0. \quad (1)$$

Note that in the proof of Lemma, without loss of generality, we assume that $I_v = \{1, 2, \ldots, d^c(v)\}$ for simply. In fact, for distinct vertices $u, v \in V(G)$, $I_u$ may be not equal to $I_v$, and may be not a subset of $[1, C(G)]$.

The *friendship graph* $F_k$ is a graph consisting of $k$ triangles sharing a common vertex. Finally, we obtain some color degree condition for the existence of some rainbow triangles sharing one common vertex, i.e., the underlying graph is a friendship subgraph. This extends Theorem 1 in another way.

**Theorem (X. Li, Ning, Shi, Zhang, Dec. 2021+)**

*Let $k \geq 2$ and $n \geq 50k^2$. Let $(G, C)$ be an edge-colored graph on $n$ vertices. If $\delta^c(G) \geq \frac{n}{2} + k - 1$ then $G$ contains $k$ rainbow triangles sharing one common vertex.*

The following conjecture is due to Hu, Li and Yang (DM, 2020). I like this conjecture very much.

### Conjecture

*Let G be an edge-colored graph on $n \geq 3k$ vertices. If $\delta^c \geq \frac{n+k}{2}$, then G contains k disjoint rainbow triangles.*

- We highly suspect the tight one is $\frac{n+1}{2}$ for $n = \Omega(k^2)$.

On the other hand, maybe an answer to the following is positive.

## Problem

*Let $n, k$ be two positive integers. Let $(G, C)$ be an edge-colored graph on $n$ vertices with $\delta^c(G) \geq \frac{n+1}{2}$. Does there exist a constant $c$, such that if $n \geq ck$ then $G$ contains a properly-colored $F_k$?*

Recall that $f(n) := \min\{rt(G) : G \in \mathcal{G}_n^*\}$. We conclude this paper with the following more feasible problem.

## Problem

*Determine the value of* $\lim\limits_{n\to\infty} \frac{f(n)}{n^2}$.

We left two major problems to the young students full of vigour.

**Problem**

*Find a good out-degree condition for disjoint directed triangles in oriented graphs.*

**Problem**

*Find a good sufficient condition for the existence of rainbow cycles in edge-colored graphs.*

**Theorem (Tomon, 2022; a corollary of Wang, 2022)**

*Suppose $G$ is a properly edge colored graph on $n$ vertices with average degree at least $(\log n)^{2+o(1)}$. Then $G$ contains a rainbow cycle.*

# Thank you for your attention!